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System Reliability of Offshore Jacket Structures by Ideal Plastic Analysis

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ABSTRACT

Recent developments in system reliability analysis of offshore jacket structures based on ideal plastic analysis methods are presented. The static theorem of ideal plasticity theory is applied in a probabilistic setting and upper and lower bounds on the system reliability are established. A comparison with the reliability for first member yielding provides a quantitative measurement of the system redundancy. The analysis is performed for static overloading and a probabilistic load model is developed. Three example structures of up to 270 members are used to demonstrate the analysis procedure.

INTRODUCTION

In present design practice attention is focused on assuring sufficient reliability of the structural elements individually rather than on assuring sufficient reliability of the structural system as a whole. However, system effects due to redundancy and many failure modes can be significant and should be taken into account in a rational design procedure. This, of course, requires operational methods for assessing the reliability of a structural system.

Different approaches can be applied for reliability analysis of structural systems with system failure involving failure of several elements of the structure. One approach is to set up an ideal plastic model of the structure and then define failure (limit state) as plastic collapse (formation of a mechanism) of this model. Another possibility is to adopt a model in which the structural elements exhibit some deformation-load effect behavior and define failure of the structure as the event of, e.g., a singular stiffness matrix or excessive deformations of the structure. A review on reliability models for structural systems is given in [1,2].

The ideal plastic model for the reliability with respect to progressive failure of a structure has been undertaken in numerous works, see [3-8] amongst others. This is mainly because such systems can be conveniently analyzed with respect to plastic collapse applying the lower and upper bound theorems of plasticity theory (theorems of limit analysis). In most reported work the analyses have been based on the upper bound theorem according to which an upper bound on the reliability can be evaluated on basis of a set of plastic mechanisms. If the set of mechanisms is complete, the upper bound coincides with the exact reliability with respect to plastic collapse. However, typically it is not practicable to take into account the complete set of mechanisms. In fact, even for simple structures the number can be infinite and the evaluation of the reliability is non-trivial. Methods for identifying a set of significant (most likely) mechanisms can be used for some types of structures. On basis of these a close upper bound on the reliability may be obtained. The formulation of the equation of virtual work for a given plastic mechanism can be rather involved, e.g. in the case of yield surfaces of random shape. The same holds if the geometry of the structure is assumed random.

In this paper recent developments for evaluating the reliability with respect to plastic collapse on basis of the lower bound theorem are presented. Both a lower and an upper bound on the reliability are obtained. The reliability model is formulated for a spatial truss structure, but the generalization to a spatial frame structure taking into account load-effect interaction in potential points of yielding is straightforward, [9,10]. The analysis procedure is demonstrated on example structures of up to 270 members.

STRUCTURAL MODEL

An n times redundant, plane or spatial truss structure of m bars is considered. The external loading is given in terms of a finite set of nodal forces $\mathbf{Q} = (Q_1, Q_2, \cdots, Q_l)$ and the normal forces in the bars are denoted by $\mathbf{N} = (N_1, N_2, \cdots, N_m)$. The bars are assumed to be ideal plastic. The yield load in tension is N_i^+ and the yield load in compression N_i^-, $i = 1,2,...,m$. The considered limit state of the structure is plastic collapse, i.e. formation of a mechanism.

LOWER BOUND THEOREM FORMULATION OF THE RELIABILITY

The lower bound theorem of limit analysis is valid for ideal plastic structures, i.e. i) the yield surface does not change during deformation, and ii) the yield surface is convex and the plastic strain rates are derivable from the yield function through the flow rule (normality condition) under the assumption that *iii*) changes in geometry of the structure at plastic collapse are insignificant. The lower bound theorem states that the structure is able to carry the external load if and only if there exists a statically admissible set of internal forces such that these nowhere violate the yield condition, see e.g. [11].

Focusing on equilibrium states of the structure it is convenient to apply a force method formulation. The complete set of statical conditions are expressed in terms of the normal forces N and the external nodal forces Q as

$$\mathbf{A}\mathbf{N} = \mathbf{Q} \tag{1}$$

A is the equilibrium matrix given in terms of the geometry of the structure. By Gauss-Jordan elimination or an equivalent procedure within the force method [12,13] the solutions to (1) can be expressed as

$$\mathbf{N} = \mathbf{B}_Q\mathbf{Q} + \mathbf{B}_Z\mathbf{z} \tag{2}$$

where $\mathbf{z} = (z_1, z_2, \cdots, z_n) \in R^n$ is called the vector of redundants. If the elimination procedure is carried out such that each z-component corresponds to a normal force in a bar of the truss structure, (2) expresses a choice of a statically determinate primary system.

The state of the ith bar is described by two functions

$$\begin{aligned} g_{2i-1}(N_i^-, N_i) &= N_i^- + N_i \\ g_{2i}(N_i^+, N_i) &= N_i^+ - N_i \end{aligned} \tag{3}$$

corresponding to yielding of the bar in compression and tension respectively. If both functions are positive the ith bar behaves elastically. The behavior of the truss structure is thus described by $2m$ functions.

The physical basic variables in the formulation of the reliability comprise the nodal forces Q, the yield forces ($\mathbf{N}^-, \mathbf{N}^+$) and a number of geometrical variables. However, throughout this presentation the geometry of the structure is assumed deterministic. If this is not the case, the equilibrium matrix in (1) is random and the analysis becomes more complicated. Let the variables $\mathbf{Q}, \mathbf{N}^-, \mathbf{N}^+$ be random and assume the joint distribution to be continuous. Then a transformation T exists such that

$$\mathbf{U} = T(\mathbf{Q}, \mathbf{N}^-, \mathbf{N}^+) \tag{4}$$

is a normalized Gaussian vector with independent components, see [14]. Let the dimension of this vector be q. The behavior of the truss structure is now described by $2m$ functions g_i in the q-dimensional u-space defined such that

$$\min_{i=1}^{2m} [g_i(\mathbf{u};\mathbf{z})] \begin{cases} > 0 & \textit{elastic behavior} \\ \leq 0 & \textit{yielding in some point} \end{cases} \tag{5}$$

The reliability $1 - p_f$ with respect to plastic collapse can then be expressed as

$$1 - p_f = P\{g(\mathbf{U}) > 0\} \tag{6}$$

where

$$g(\mathbf{u}) = \max_{\mathbf{z} \in R^n} [\min_{i=1}^{2m} [g_i(\mathbf{u};\mathbf{z})]] \tag{7}$$

The max-operation expresses that an admissible equilibrium distribution of internal forces is sought for each set of values of the basic variables. The system representation as given by (7) may obviously be referred to as a parallel system with an infinity of series subsystems.

For some of the considerations in the following it is necessary to recast the expression for the reliability. Noting that U can be expressed as $\mathbf{U} = R\mathbf{A}$ ($R \geq 0$) where R^2 is a chi-square distributed random variable with q degrees of freedom and A is a q-dimensional unit vector uniformly distributed on the unit sphere Ω_q the reliability can be given in the form

$$\begin{aligned} 1 - p_f &= \int_{\Omega_q} P\{g(R\mathbf{A}) > 0 \mid \mathbf{A} = \mathbf{a}\} f_{\mathbf{A}}(\mathbf{a}) d\mathbf{a} \\ &= \int_{\Omega_q} P\{g(R\mathbf{a}) > 0\} f_{\mathbf{A}}(\mathbf{a}) d\mathbf{a} \end{aligned} \tag{8}$$

where $f_{\mathbf{A}}(\mathbf{a}) = \textit{constant}$ is the uniform density on the unit sphere. If the safe set in u-space is star-shaped (e.g. convex) with respect to the origin, $P\{g(R\mathbf{a}) > 0\}$ is given in terms of the chi-square distribution function χ_q^2 as

$$P\{g(R\mathbf{a}) > 0\} = \chi_q^2(r(\mathbf{a})^2) \tag{9}$$

where $r(\mathbf{a})$ is the distance from the origin in u-space to the limit state surface in the direction defined by a. For fixed a, $r = r(\mathbf{a})$ is part of the solution to the optimization problem:

Determine $r \geq 0$ and $\mathbf{z} \in R^n$ such that r is maximized subject to $\min_{i=1}^{2m} [g_i(r\mathbf{a};\mathbf{z})] > 0$.

If the transformation T is linear (normally distributed physical basic variables), the optimization problem reduces to a linear programming problem.

EVALUATION OF THE RELIABILITY

The evaluation of the reliability given by (6) or (8) is non-trivial for problems of high dimensionality. The following outlines how lower and upper bounds on the reliability can be calculated under certain assumptions. First and second order reliability methods FORM/SORM as well as Monte Carlo simulation methods are applied in the calculation. Besides the reliability measures, a FORM/SORM analysis directly provides parametric sensitivity measures for the reliability with respect to deterministic and distributional parameters, see [14]. These measures can be used in a search for an optimal design.

Lower Bound on the Reliability

If the considered equilibrium distributions of internal forces is restricted by substituting $\mathbf{z} \in R^n$ in (7) by a finite number of vectors $\mathbf{z}_1, \mathbf{z}_2, \cdots, \mathbf{z}_p$ the right side of (6) provides a lower bound on the reliability. The [illegible] z-vector, z_0. The [illegible] lower bound can be optimized by [illegible] optimization problem:

> Determine [illegible] such that the probability $P\{$[illegible]$\}$ is maximized.

The amount of [illegible] can be reduced if the lower bound is [illegible] maximized by solving, in stead, the optimization problem:

> Determine $z_0 \in R^n$ such that $\min_{i=1}^{2m}[P\{g_i(U;z_0 > 0\}]$ is maximized.

If the transformation T is linear, this problem is a linear programming problem and a solution can be found efficiently. However, the lower bound on the reliability obtained this way in general turns out to be considerably smaller than the exact reliability. Moreover, the result depends strongly on the actual choice of the redundants, i.e. on the statically determinate system (B_Q and B_Z in (2)).

In [15,16] methods for improving the lower bound by considering more than one set of values for the redundant forces or by considering more than one choice of the redundants are given. The improvements of the lower bound obtained by these approaches can be significant. However, no general and efficient procedure to assure this has been reported.

Here, improvements of the lower bound on the reliability by taking the set of redundants z as a random vector Z are considered. For any outcome **u** of U a value of z can be determined as the solution to the right hand side of (7). Let this solution be denoted by **h(u)**. The reliability can thus be written as the reliability of a series system as

$$1 - p_f = P\{\min_{i=1}^{2m}[g_i(U;Z)] > 0\} \qquad (10)$$

where

$$Z = h(U) = h(T(Q,N^-,N^+)) \qquad (11)$$

Of course, the function **h** is not known. However, the right hand side of (10) provides a lower bound on the reliability for any choice of the function **h**. Two functions are applied here.

First, Z is chosen as a linear function of the external loads Q, i.e.

$$Z = xQ + z_0 \qquad (12)$$

and the lower bound based on this random z-vector is sought maximized by solving the optimization problem:

> Determine x, z_0 such that $\min_{i=1}^{2m}[P\{g_i(U;Z) > 0\}]$ is maximized.

This non-linear optimization problem is solved by sequential linear programming. The solution has the advantage of being independent of the chosen redundants (see (2)).

Secondly, Z is chosen as a linear function of the external loads Q as well as the yield forces of the bars (N^-,N^+),

$$Z = xQ + y\begin{bmatrix}N^-\\N^+\end{bmatrix} + z_0 \qquad (13)$$

and the optimization problem maximizing $\min_{i=1}^{2m}[P\{g_i(U;Z) > 0\}]$ is solved by sequential linear programming. This case includes the optimization problem formulation with deterministic safety margins in [7,17]. For the choice of Z in (13) also the following optimization problem is considered:

> Determine x, y, z_0 such that $P\{\min_{i=1}^{2m}[g_i(U,Z)] > 0\}$ is minimized.

The problem is solved by a steepest ascent method. To improve the calculation efficiency the partial derivatives of the reliability with respect to the parameters x, y, z_0 are approximated by the asymptotic results for parametric sensitivity measures known from first and second order reliability methods, see [14].

In a later section lower bounds on the reliability with respect to plastic collapse are calculated for three truss structure examples. The results are compared with the lower bound obtained by assuming a linear elastic distribution of the internal forces in the structure.

Upper Bound on the Reliability

Two methods of plastic upper bound analysis are considered, namely the *directional search method* [9,10], and the *linear combination method* [7,17].

An approximate evaluation of the reliability in (6) can be carried out by a first or second order reliability method provided the most likely failure points have been identified. Restricting the considerations to the cases where the transformation T is linear, the safe set in u-space is a polyhedral convex set.

A close upper bound on the reliability corresponding to this set can be obtained by applying only the hyperplanes defining the faces of this polyhedral set with smallest distance to the origin of u-space. From experience it is known that within plastic system reliability analysis it is often necessary to apply several hyperplanes in order to get a close upper bound. Each hyperplane can be interpreted as representing a failure mode (plastic mechanism) of the structure.

The crucial point in calculating a close upper bound on the reliability is thus to identify the significant hyperplanes. One possibility is to apply the *directional search method* [9], describing the limit state surface in u-space by $r = r(\mathbf{a})$. A starting unit vector $\mathbf{a}^0$ is chosen. The distance to the limit state surface $r^0 = r(\mathbf{a}^0)$ is determined as the solution to a linear programming problem (see end of previous section). Moreover, the unit normal vector α_1 to the limit state surface in $\mathbf{u}^1 = r^0\mathbf{a}^0$ is determined numerically. (Of course, attention should be paid to the possibility of having identified a singular point on the limit state surface). The safety margin corresponding to the face of the safe set in this point is then determined by the reliability index $\beta_1 = \alpha_1^T a^0 r^0$ and the unit normal vector α_1. With the new starting vector $a^1 = \alpha_1$ the same procedure is repeated resulting in $\beta_2 = \alpha_2^T \alpha_1 r^1$. The procedure is continued until a stop criterion ($\alpha_i^T \alpha_{i+1} \approx 1$) is fulfilled. The same scheme may now be repeated with a new starting vector a^0. In each step of the algorithm the result (β_i, α_i) is stored if the safety margin is not highly correlated ($\rho_{ij} = \alpha_i^T \alpha_j \approx 1$) with a previously identified safety margin.

Each sequence in the procedure is similar to the well-known Rackwitz-Fiessler algorithm of identifying a first order reliability index, see [14]. However, deviations are, that only points on the limit state surface are considered (due to the formulation in $r(a)$), and that the result of each step in the algorithm is stored. The procedure is stopped by some convergence criterion based on the probability content of the identified polyhedral set. This probability is given by the multi-variate normal distribution function and can be evaluated approximately, e.g. in terms of upper and lower bounds [7,12]. The starting vector a^0 in each sequence of the procedure is generated by simulation using a sampling density giving preference to directions corresponding to a lower fractile for a resistance variable and an upper fractile for a load variable. Alternatively, the starting vectors could be generated by some deterministic procedure, e.g. producing more or less uniformly spaced points on the unit sphere in u-space.

Another plasticity theoretical way of establishing an upper bound on the reliability is by the method of linear combination of lower bound safety margins [7,17]. This method, referred to as the *linear combination method*, can briefly be outlined as follows.

Consider a linear combination of lower bound safety margins from (5) of the form

$$L = \sum_{i=1}^{m} \gamma_i \, g_{I(i)}(U; z) \tag{14}$$

where the index function $I(i)$ is equal to either $2i-1$ (compression safety margin for member i) or $2i$ (tension safety margin for member i). It can then be shown that L is an upper bound safety margin corresponding to a plastic mechanism if i) L is independent of z, and ii) the coefficients γ_i all are non-negative, [17]. The linear combination method uses this fact by establishing upper bound safety margins as linear combinations of the form in (14) using so-called *dominant* lower bound safety margins. The dominant lower bound safety margins are defined as follows.

A plastic lower bound analysis is performed considering deterministic redundants z_0. To each of the $2m$ lower bound safety margins in (5) a reliability index $\beta_j = \Phi^{-1}(P[g_j(u, z_0) > 0])$ is associated. The lower bound is sought maximized by maximizing the smallest reliability index with respect to the deterministic redundants. The safety margins with reliability index equal to the smallest value (the Hasofer Lind reliability index) in the solution are the dominant lower bound safety margins. It can be shown that for normally distributed basic variables a value of the redundants exist such that there are at least $n+1$ dominant lower bound safety margins [17].

Significant upper bound safety margins are searched by considering linear combinations (primarily) of the dominant lower bound safety margins. In other words, the dominant safety margins are taken as indicators for which members in the structure are likely to yield under plastic collapse. As opposed to failure tree reliability analyses based on successive elastic analyses, a *plastic analysis* is here used to identify members that are likely to be yielding in a significant mechanism. Details about the strategy for combining the lower bound safety margins can be found in [7,17].

Reliability Calculation by Simulation

The reliability $1-p_f$ or the probability of failure p_f can be estimated by Monte Carlo simulation. In particular the method of directional simulation [18,19] seems appropriate using the expression for the reliability in (8). The simulation is carried out by generating N outcomes $a_1, a_2, \cdots, a_i, \cdots, a_N$ of the unit vector A. For each outcome a_i the distance $r(a_i)$ is determined by solution of an optimization problem. With

$$p_i = 1 - \chi_q^2(r(a_i)^2) \tag{15}$$

p_f is estimated by

$$\hat{E}[P] = \frac{1}{N} \sum_{i=1}^{N} p_i \tag{16}$$

and an estimate on the variance on the estimator is

$$\hat{Var}[P] = \frac{1}{N(N-1)} \sum_{i=1}^{N} (p_i - \hat{E}[P])^2 \tag{17}$$

Results from this type of simulation can be found in the examples in a later section. In [19] other examples are given and a method of reducing the variance of the estimator by importance sampling is proposed. Furthermore it is shown how parametric sensitivity measures can be simulated.

APPLICATION TO OFFSHORE JACKET STRUCTURES

Reliability models of the type considered in this paper are used to evaluate a reliability measure for jacket-type offshore structures under extreme environmental loading [8,20]. In this context a computer analysis program RAPJAC (Reliability Analysis of Plastic Jackets) based on the reliability methods presented in the preceding sections has been developed, [21,22].

The program can be used within the SESAM* program system for structural analysis. In particular, the SESAM preprocessor for generating geometry and topology of a structure as well as a utility program for calculating water particle kinematics can be applied in connection with the reliability analysis program. Of course, links to other commercial general purpose structural analysis program systems are straightforward to implement.

At present a first version of RAPJAC has been completed. The version handles spatial truss structures. The basic assumptions for the reliability analysis are: *i*) the overall geometry of the structure is assumed deterministic, and *ii*) the basic variables (yield forces of bars and nodal forces) are jointly normal. The latter assumption can be relaxed, since the present reliability analysis program can be coupled to general purpose probabilistic analysis programs handling any type of distributions like the PROBabilistic ANalysis Program PROBAN from A.S Veritas Research, [23]. In the reliability model all uncertainty is described by random variables and the program is aimed at reliability assessment with respect to an instantaneous overload.

An automatic generation of nodal forces from waves and current is available. The forces are established on basis of a deterministic water particle velocity field calculated (under the assumption that the structure is absent) from a discrete, plane wave applying a specified wave theory (Airy, Stokes *5th* order, Deans Stream Function or the Cnoidal wave theory) and using specified values for the wave height, period and direction as well as current velocity and direction. Given such a velocity field the joint distribution for the nodal forces is found using the Morison formula as follows.

Consider a point on the axis of a tubular member below the water surface. Let the water particle velocity in the point be $\mathbf{v} = (v_1, v_2, v_3)$ where the first two components are mutually orthogonal and orthogonal to the member axis, and the third component is parallel to it. Let the wave force intensity per unit length of the tubular member at the considered point be $\mathbf{q} = (q_1, q_2, q_3)$, where q_1 and q_2 are orthogonal to the member axis and q_3 is parallel to it. $\mathbf{q}$ is then assumed given by

$$\begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \frac{1}{2}\rho_W (D+2H) \begin{bmatrix} C_{D,11} & C_{D,12} & 0 \\ C_{D,21} & C_{D,22} & 0 \\ 0 & 0 & C_{D,33} \end{bmatrix} \begin{bmatrix} v_1\sqrt{v_1^2+v_2^2} \\ v_2\sqrt{v_1^2+v_2^2} \\ v_3|v_3| \end{bmatrix}$$

$$+ \frac{\pi}{4}\rho_W (D+2H)^2 \begin{bmatrix} C_{M,11} & C_{M,12} & 0 \\ C_{M,21} & C_{M,22} & 0 \\ 0 & 0 & C_{M,33} \end{bmatrix} \begin{bmatrix} \dot{v}_1 \\ \dot{v}_2 \\ \dot{v}_3 \end{bmatrix} \quad (18)$$

where ρ_W is the specific mass of the water, D is the diameter of the tubular member at the considered point, H is the excess radius of the member due to marine growth, the C_D parameters are drag coefficients and the C_M parameters are inertia coefficients. For the sake of simplicity the off-diagonal elements as well as the third diagonal element in the two matrices are here taken as zero. Furthermore, the two diagonal elements in each matrix are assumed equal to C_D and C_M, respectively, since the structural member is tubular. However, it is noted that the more general formula in (18) is implemented in RAPJAC.

The coefficients C_D and C_M are assumed to be spatial Gaussian white noise processes over the structure. In a point, the two processes may be correlated. Typically negative correlation originating from statistical uncertainty is assumed. The white noise assumption has been introduced in order to reduce the computational effort when integrating the force intensities into nodal forces for which the second moment representation must be computed. Furthermore, the excess thickness H can be assumed to be a spatial white noise process. In this is the case, the second moment representation for the force intensities $\mathbf{q}$ is found approximately by a second order expansion in the mean point. This approximation is not fully consistent with a modern FORM/SORM analysis but has been introduced to reduce the computational effort, and to maintain the assumption of normally distributed nodal forces. The approximation can be avoided by coupling to PROBAN, but in that case the number of basic variables increases radically. In summary, the uncertainty in the wave and current forces is modeled by random coefficients in the Morison equation together with a random diameter of the member. The mean value and standard deviation of the random variables at a given position can be specified as a piece-wise linear function of the elevation above the sea bed.

Gravity and buoyancy loads on the truss structure are generated automatically as well. Loads on the deck structure, e.g. gravity loads, live loads and wind loads can not be generated by the present program but load models in terms of nodal forces on the truss structure must be set up by the user.

The program provides lower and upper bounds on the reliability with respect to plastic collapse. A lower bound can be established as the reliability with respect to initial yielding under the assumption of elastic force distribution in the structure. Furthermore, the plasticity theoretical lower bounds can be calculated. The upper bound analysis is based on two different and completely independent plasticity theoretical approaches comprising the directional search procedure and the method of combining lower bound safety margins. Finally, the reliability can be checked by directional simulation. Results from such simulations are given as an estimate on the reliability together with an estimated coefficient of variation on the estimator.

*) SESAM is a trademark of A.S. Veritec Sesam Systems, Norway.

The program has been applied for research purposes. It is planned to be available for practical purposes like comparisons between alternative design solutions, evaluation of the reliability of damaged structures, determination of the importance of the structural members and for identification of an optimal design under a complete load description. It is furthermore the intention to continue the implementation of reliability methods for offshore jacket structures to the extent where real life sized structures modeled as spatial frameworks can be handled.

EXAMPLES

Plane truss structure of 10 members

An $n=2$ times redundant plane truss structure of $m=10$ bars is considered, Fig. 1. The yield forces of the 10 bars are assumed to be normally distributed with the following representation:

$$E[N_i^-] = \begin{cases} 0.2\mu_N & \text{for } i = 1,2,3,4 \\ 1.0\mu_N & \text{for } i = 5,6 \\ 0.8\mu_N & \text{for } i = 7,8,9,10 \end{cases}$$

$$E[N_i^+] = \begin{cases} 0.4\mu_N & \text{for } i = 1,2,3,4 \\ 1.6\mu_N & \text{for } i = 5,6,7,8,9,10 \end{cases}$$

$$\frac{D[N_i^-]}{E[N_i^-]} = \frac{D[N_i^+]}{E[N_i^+]} = 0.15$$

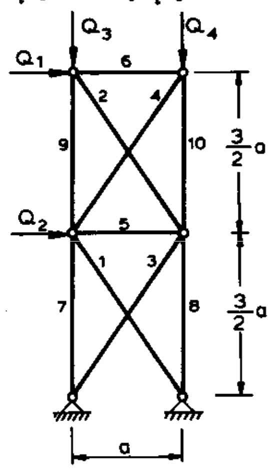

Fig. 1. Plane truss structure.

All variables are assumed independent, except N_i^- and N_i^+, for which $\rho[N_i^-,N_i^+]=0.90$ for all i. The stiffness of the bars no. 1-4 are assumed to be the equal and 20% of the stiffness of the remaining bars.

Three loading cases for the structure are considered. In load case I only the horizontal force Q_1 is subjected, in load case II both horizontal forces Q_1 and Q_2 are subjected, and in load case III the two vertical loads Q_3 and Q_4 are acting. The representation of the load variables are

$$E[Q_1]=0.10\mu_N \ , \ D[Q_1]=0.03\mu_N$$
$$E[Q_2]=0.05\mu_N \ , \ D[Q_2]=0.01\mu_N$$
$$E[Q_3]=0.50\mu_N \ , \ D[Q_3]=0.04\mu_N$$
$$E[Q_4]=0.50\mu_N \ , \ D[Q_4]=0.04\mu_N$$

The correlation coefficient between Q_1 and Q_2 is assumed to be 0.8, and the correlation coefficient between Q_3 and Q_4 is assumed to be 0.5.

The reliability of the truss structure with respect to plastic collapse is calculated by the different methods. The results of the lower and upper bound analysis are given in Table 1, 2, 3 and 4. The exact reliability for this (small) structure can be found by considering all plastic mechanisms, and for all three load cases the upper bound result coincides with the exact result.

The results of the lower bound analysis are given in terms of the Hasofer-Lind reliability index β_{HL} (equal to the reliability index corresponding to the reliability $\min_{i=1}^{2m}[P\{g_i(U;Z)>0\}]$) and the system reliability index β_G. The only lower bounds close to the exact result are those obtained by optimizing the system reliability index. The elastic lower bound as well as the plastic lower bounds based on linear programming turn out to be significantly smaller than the exact reliability.

In Tables 2, 3 and 4 results are given for the two independent upper bound methods. The methods yield the same and exact result on the system reliability index in all three load cases. Experience from other examples as well indicates that the upper bound analyses typically give close upper bounds, implying that the methods identify all significant mechanisms.

Table 1. Results of lower bound analysis.
The numbers in parenthesis are the numbers of
iterations in the non-linear programming problems.

Load case	I $[Q_1]$		II $[Q_1,Q_2]$		III $[Q_3,Q_4]$	
Type of analysis	β_{HL}	β_G	β_{HL}	β_G	β_{HL}	β_G
Elastic lower bound:	2.67	2.47	1.41	1.39	2.78	2.36
Plastic lower bound:						
max β_{HL} w.r.t. z	3.79	3.37	2.72	2.33	3.08	2.43
max β_{HL} w.r.t. x,z	3.91	3.55 (66)	2.91	2.54 (19)	3.10	2.45 (12)
max β_{HL} w.r.t. x,y,z	3.98	3.59 (19)	2.91	2.54 (8)	3.28	2.64 (40)
max β_G w.r.t. x,y,z	4.09	3.96 (219)	3.32	3.11 (117)	3.19	2.74 (64)
Exact result		4.34		3.36		3.20

Table 2. Upper bound analysis for case I.			
Mech. no.	Linear combinations	Directional search	Bars in yielding
1	4.38	4.38	$1^- \ 8^-$
2	4.87	4.87	$2^- \ 4^+$
3	4.87	4.87	$1^- \ 3^+$
4	6.25		$3^- \ 10^-$
5		6.25	$3^+ \ 7^+$
6	6.49	6.49	$7^+ \ 10^-$
7	7.15	7.15	$4^+ \ 8^+$
8	8.22		$4^+ \ 6^+ \ 10^-$
9	8.55		$1^- \ 7^-$
β_G	4.34	4.34	

Table 3. Upper bound analysis for case II.			
Mech. no.	Linear combinations	Directional search	Bars in yielding
1	3.44	3.44	$1^- \ 3^+$
2	3.71	3.71	$1^- \ 10^-$
3	4.87	4.87	$2^- \ 4^+$
4	5.87	5.87	$3^+ \ 7^+$
5		6.13	$7^+ \ 10^-$
6	7.15		$4^+ \ 8^+$
7	7.88		$3^+ \ 4^+$
8	8.36		$3^+ \ 10^+$
9	8.55		$1^- \ 7^-$
β_G	3.36	3.36	

Table 4. Upper bound analysis for case III.			
Mech. no.	Linear combinations	Directional search	Bars in yielding
1	3.62	3.62	$2^- \ 8^-$
2	3.62	3.62	$4^- \ 8^-$
3	3.62	3.62	$1^- \ 7^-$
4	3.62	3.62	$3^- \ 7^-$
5	3.76	3.76	$1^- \ 10^-$
6	3.76	3.76	$4^- \ 9^-$
7	3.76	3.76	$3^- \ 10^-$
8	3.76	3.76	$2^- \ 9^-$
9	7.08	7.08	$5^+ \ 8^-$
10	7.39		$6^+ \ 7^- \ 8^-$
11		7.45	$6^+ \ 7^- \ 9^-$
12		7.88	$3^+ \ 6^+ \ 8^-$
13		7.91	$4^+ \ 6^+ \ 10^-$
14	8.92		$7^- \ 10^+$
15	8.92		$8^- \ 9^+$
β_G	3.20	3.20	

Spatial truss structure of 48 members

The model of a steel jacket offshore platform in Fig. 2 is considered. The structure is an $n=12$ times redundant spatial truss tower with $m=48$ tubular bars. All geometry variables, i.e. the dimensions of the structure given in Fig. 2 and the dimensions of the bars given in Table 5 are assumed to be deterministic.

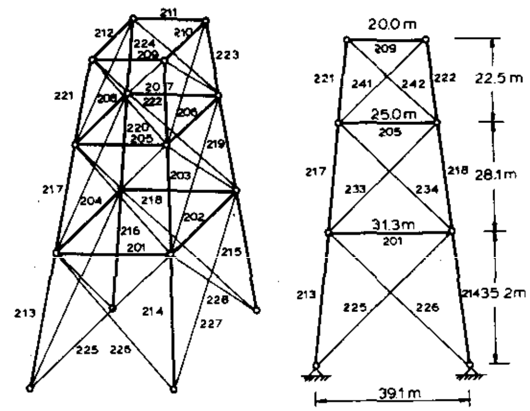

Fig. 2. Spatial truss tower of 48 tubular bars.

Table 5. Dimensions and mean tension yield forces of bars. The ratio between the diameter d and the wall thickness t is assumed to be $d/t = 60$ for all bars.			
Bar No.	d $[m]$	Area $[m^2]$	$E[N_i^+]$ $[MN]$
201-204	2.0	0.210	67.20
205-208	1.5	0.116	37.07
209-212	1.0	0.053	16.80
213-224	2.5	0.324	103.78
225-232	1.5	0.116	37.07
233-248	1.2	0.074	23.73

The yield forces and loads are assumed to be normally distributed with the following representation:

$$E[N_i^-] = 0.75E[N_i^+] \quad \textit{given in Table 5}$$

$$D[N_i^-] = 0.15E[N_i^-], \ D[N_i^+] = 0.10E[N_i^+]$$

Furthermore it is assumed that all yield forces are equi-correlated with correlation coefficient 0.5.

The external loading on the structure is due to gravity, live load, wind, wave and buoyancy. The following load models are applied:

Gravity and live loads: Gravity and live loads from the deck structure are modeled by four vertical loads, one in each of the four top level nodes of the truss structure. Each force has a mean value 20 MN, a coefficient of variation 0.10, and the four forces are equi-correlated with correlation coefficient 0.5. Gravity loads of the truss structure itself are referred to the nodes as single forces, and are calculated for a specific mass of the tubular members of

$7.85 \cdot 10^3\, kg/m^3$. Furthermore, additional gravity loads are included (e.g. from inside stiffeners in the members and in the joints) by assuming that the specific mass of the interior of the members is $0.25 \cdot 10^3 kg/m^3$. The gravity forces on the truss structure are assumed deterministic.

Wind load: Wind load on the deck structure is modeled by a horizontal and a vertical force in each of the four top level nodes. The magnitude of these forces are all assumed proportional to a random variable of mean $1.0\, MN$ and with coefficient of variation 0.30. The direction of the wind model forces and the coefficients of proportionality are given in Fig. 3. The model is based on the assumption that the wind acts in a direction of 30 degrees with one side of the truss structure. Wind loading on the jacket structure itself is neglected.

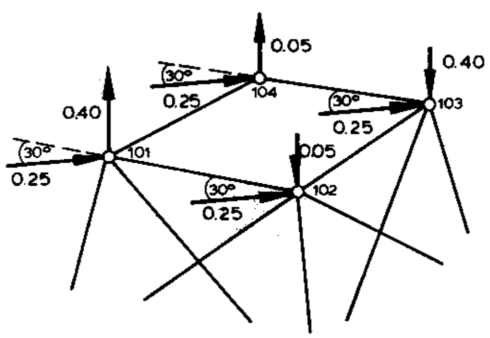

Fig. 3. Illustration of direction and magnitude of nodal forces from the wind loading on the deck structure.

Wave and buoyancy loads: The marine loading is calculated on basis of the water particle kinematics for a $5th$ order Stokes theory wave of height $h = 25m$ and wave period $T = 17s$. The water depth is assumed to be $d = 70m$, Fig. 4, and the direction of the wave is the same as the direction of the wind, Fig. 5. It is assumed that no current is present. A second moment representation for the nodal forces of the marine loading is determined under the assumptions that the drag coefficient C_D and the inertia coefficient C_M in the Morison formula as well as the excess thickness of the tube walls due to marine growth H are spatial Gaussian white noise processes. The mean values and standard deviations as function of the position are given in Table 6. Finally, vertical deterministic buoyancy loads on the member parts under the sea surface are calculated and added to the respective nodal forces.

The position of the wave is defined by the wave phase angle θ. For $\theta = 0^\circ$ the wave crest is above the first support of the structure (in the direction of the wave propagation). For θ approximately equal to 20°, the wave crest is in the middle of the structure.

An elastic lower bound reliability analysis and a plastic upper bound analysis based on the method of linear combination of lower bound safety margins are carried out for different values of the wave phase angle θ. Selected cases have been checked by directional simulation. The results are shown in Fig. 5.

The difference between the reliability with respect to initial failure and the reliability with respect to total plastic collapse is seen. For different positions of the wave, different failure modes are dominating. The same holds for the most likely member to yield in the lower bound analysis. Only a small difference between the reliability of the most likely element to yield and yielding in any member is observed in the extreme loading situation. This is due to high correlation between element safety margins in this case. The same tendency is observed in the upper bound analysis with respect to plastic collapse, where the reliability index for the most likely mechanism is only slightly higher than the plastic system reliability index.

Finally it is noted that the variation in θ of the reliability index with respect to plastic collapse in this case follows closely the variation of the elastic system reliability index.

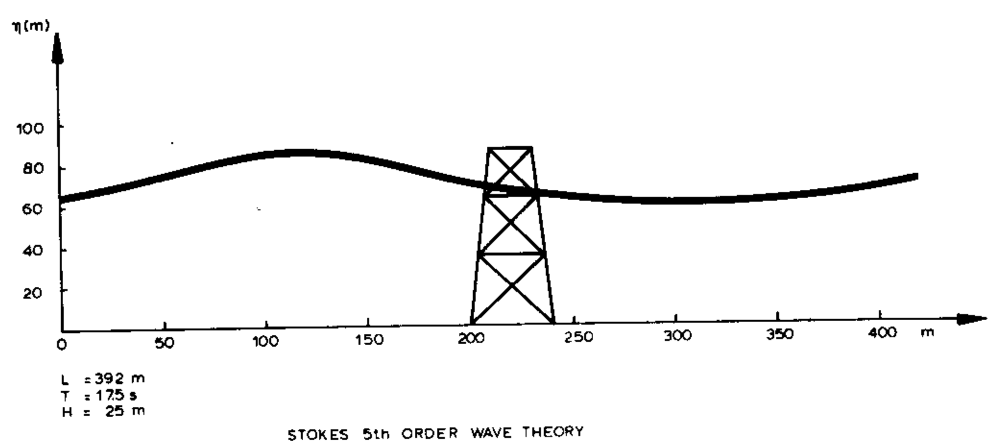

Fig. 4. Illustration of truss structure and the example wave.

Table 6. Mean value and standard deviation of C_D, C_M and H. Coefficients are a function of elevation above sea bed. Between the given levels the quantities vary linearly, and above 75m they are constant. At a given position the three variables are assumed uncorrelated.

Elevation z	$E[C_D]$	$D[C_D]$	$E[C_M]$	$D[C_M]$	$E[H]$	$D[H]$
00 m	1.0	0.4	2.0	0.3	0.00 m	0.000 m
30 m	1.0	0.4	2.0	0.3	0.01 m	0.002 m
65 m	1.0	0.4	2.0	0.3	0.05 m	0.010 m
75 m	1.0	0.4	2.0	0.3	0.10 m	0.030 m

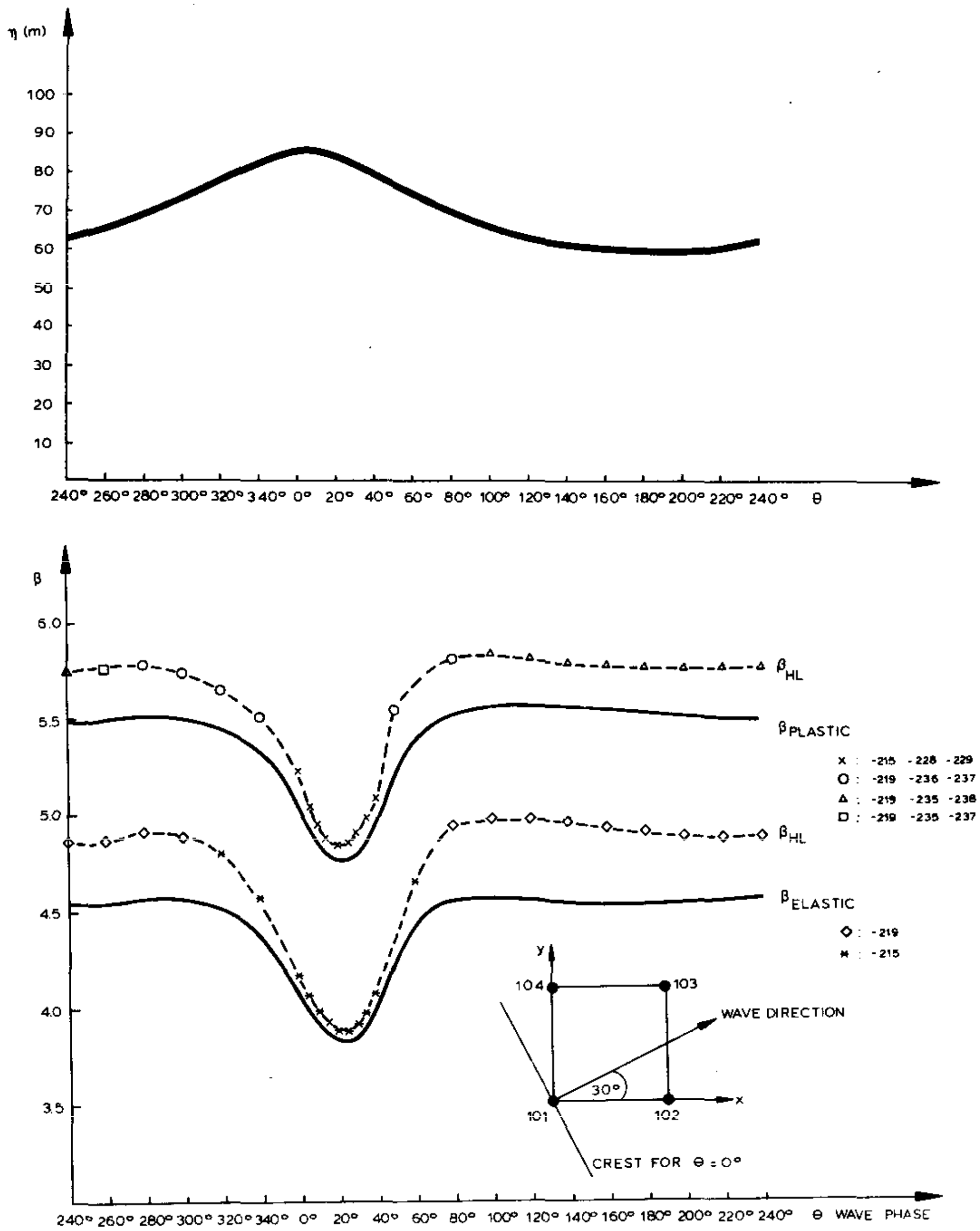

Fig. 5. Results of elastic reliability analysis with respect to initial yielding and plastic upper bound reliability analysis with respect to plastic collapse for different positions of the wave crest.

Spatial truss structure of 270 members

An $n=108$ times redundant spatial truss structure model of a jacket structure is considered, see Fig. 6. The structure has 6 topologically identical levels, each made up of 9 upright bars, 12 horizontal bars and 24 diagonal bars, i.e. $m=270$. The dimensions of the structure as well as the diameters of the bars are given in the figure. The bars have a diameter/wall thickness ratio of 60. All dimensions are assumed to be deterministic.

The structure is subjected to dead load, live load, wind load and wave load. In each of the 9 nodes in the top level a dead load G_i and a live load P_i ($i=1,2,...,9$) are acting. In the same nodes a horizontal wind load with magnitude V acts parallel to a side of the structure. These forces represent the loading on the jacket from the deck structure. Finally, a set of forces representing the loading from wave and current are assumed in the 9 nodes of each of the 5 remaining levels. These 45 forces are all assumed proportional to a random quantity W. The factor of proportionality is constant within a level and the values are given in Fig. 6. It is noted that the wave loading in this case is not generated automatically but an example load has been assumed.

The 560 basic variables N_i^-, N_i^+, $G_1, G_2, \ldots, G_9$, $P_1, P_2, \ldots, P_9$, V, and W are assumed jointly normally distributed with the following representation:

$E[N_i^-]=A_i\,256MPa$, $D[N_i^-]=0.15E[N_i^+]$ for $i=1,2,...,270$
$E[N_i^+]=A_i\,320MPa$, $D[N_i^+]=0.10E[N_i^+]$ for $i=1,2,...,270$
$\rho[N_i^-,N_i^+]=0.8$ for $i=1,2,...,270$
$\rho[N_i^k,N_j^l]=0.4$ for $i \neq j$ and $k,l=-,+$

$E[G_i]=7.5MN$, $D[G_i]=0.10E[G_i]$ for $i=1,2,...,9$
$E[P_i]=2.5MN$, $D[P_i]=0.20E[P_i]$ for $i=1,2,...,9$
$E[V]=0.5MN$, $D[V]=0.25E[V]$
$E[W]=2.0MN$, $D[W]=0.30E[W]$
$\rho[G_i,G_j]=0.5$, $\rho[P_i,P_j]=0.7$ for $i \neq j$
$\rho[V,W]=0.9$

where A_i denotes the cross sectional area of the i th bar. The correlation coefficients not given above are assumed to be zero.

The following results are obtained:

Elastic lower bound analysis:
Element reliability: $\beta_{HL}=3.71$
System reliability : $\beta_G=3.00$

Plastic lower bound analysis with deterministic redundants:
Element reliability: $\beta_{HL}=4.38$
System reliability : $\beta_G=3.20$

Plastic upper bound analysis:
Number of identified mechanisms: 168
3 most likely mechanisms : $\beta=6.10$, 6.10 and 6.11
System reliability based on identified mechanisms: $5.84<\beta_G<5.86$

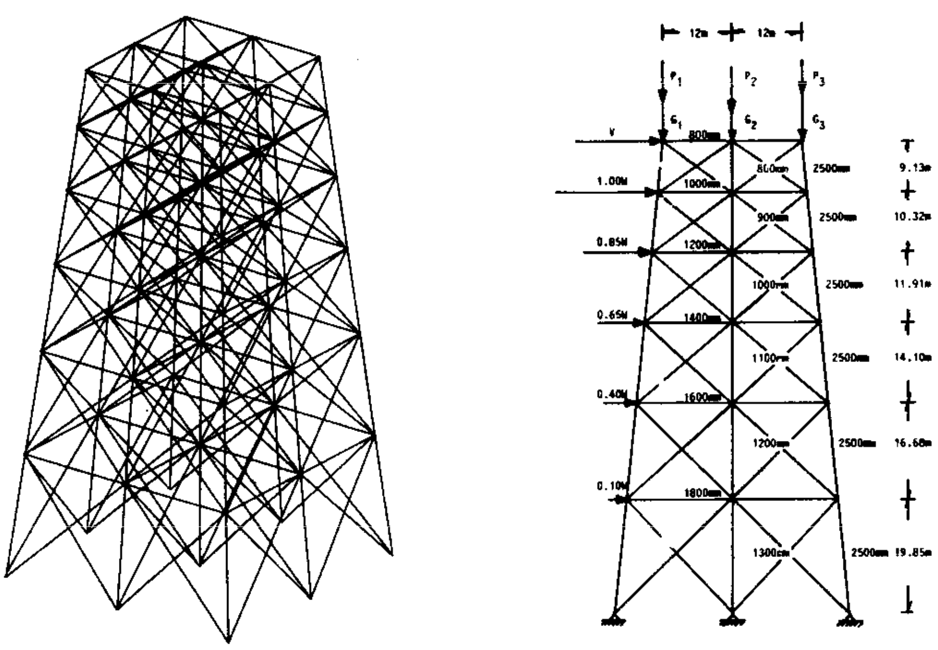

Fig. 6. Spatial truss structure with 270 tubular bars.

The following remarks to the results are appropriate. No efforts have been used to optimize the plastic lower bound. The result shown is the one obtained in the initialization of the method of combining lower bound safety margins. In this case only a moderate increase is obtained in the lower bound on the reliability index as compared to the result based on an elastic analysis.

The 3 most likely mechanisms involve yielding in bars belonging to the third level (from the bottom) of the structure. In fact, each mechanism corresponds to failure of the four diagonals in one of the three vertical planes parallel to the direction of the wave and wind load. Also mechanisms involving yielding in a rather high number of elements have been identified. As an example, a mechanism with reliability index 6.52 involves yielding in 13 bars in the first level and 6 bars in the second level. Among the identified mechanisms the maximum number of yielding bars in one mechanism is 21.

Finally, a directional simulation of the reliability has been performed. The identified mechanisms have been applied to construct an importance sampling density. This density is used in 50% of the simulations, whereas uniform sampling on the unit sphere is applied in the remaining simulations. A sample of 1000 simulations has been generated. In this example the dimension of the u-space is as high as 560. The computation of the distance to the limit state surface is quite time consuming since a linear programming problem with 540 constraints (2 times the number of bars) and 227 variables (2 times the degree of redundancy plus the distance) must be solved. Furthermore, the high dimension of u-space as well as the relatively small probability of failure implies that even though the simulation results in Table 7 shows a very small coefficient of variation, the sample size may be too small to guarantee that all significant mechanisms have been found. However, the simulation does not show that the opposite is the case.

Table 7. Directional simulation of the reliability with respect to plastic collapse using 50% importance sampling and 50% uniform sampling.

Sample size N	Estimated reliability index	Estimated coefficient of variation on the probability of failure
10	5.86	0.36
20	5.84	0.24
50	5.82	0.13
100	5.84	0.10
200	5.84	0.07
500	5.84	0.06
1000	5.84	0.04

CONCLUSION

Recent developments for evaluation of the system reliability with respect to plastic collapse based on the lower bound theorem are presented, and on basis of these a lower and an upper bound on the reliability are obtained. The reliability model is formulated for spatial truss structures. A program for plastic reliability analysis of offshore jacket structures has been developed. The program can be applied within a larger commercial program package for structural analysis.

In brief, the basic variables that specifically can be modeled as random within the program are:

- compression and tension yield forces of bars (uncertainties e.g. in yield stress and cross sectional area, as well as model uncertainty),
- nodal forces describing external loadings such as dead load, live load, wind load and wave and current load,
- parameters in the Morison equation, i.e. the normal and longitudinal drag and inertia coefficients (when wave and current forces are generated automatically),
- marine growth thickness (when wave and current forces are generated automatically),
- buoyancy and gravity forces, and
- model uncertainty in wave and current forces.

The following conclusions can be drawn:

- The reliability methods for plastic systems provide a means of quantifying redundancy of structures. In common design practice such system effects are not accounted for. Furthermore, the system reliability method can e.g. be applied for evaluation of the reliability of damaged structures, and for development of reliability based optimal design.
- The upper bound on the reliability determined by a first order reliability method converges towards the exact reliability for increasing amount of calculation. Often a close upper bound on the reliability can be established with a manageable calculational effort even for real life sized structures.
- Simplifications must be introduced to make the calculation of a maximized lower bound on the reliability by a first or second order method practicable. Here, lower bounds based on one vector of the redundants are considered and three cases are undertaken: The vector of redundants is 1) deterministic, 2) linear in the nodal forces, and 3) linear in the nodal forces and the yield forces. The lower bounds resulting from an optimization of these linear combinations do not in general

converge towards the exact reliability for increasing calculation power. Results from random redundants can be significantly closer to the exact reliability than results from deterministic redundants. Typically, however, the lower bounds turn out to be somewhat smaller than the exact reliability, at least for a practicable amount of calculation.

- The method of directional simulation provides a general and rather efficient means of establishing a confidence interval on the desired reliability.
- The plastic reliability methods considered herein are valid for truss structures under the assumptions that *i*) the geometry is assumed deterministic, and *ii*) the basic problem variables are normally distributed. The formulation of the reliability can be directly generalized to frame structures with load-effect interaction and the distributional assumptions above can be relaxed. However, the calculation methods based on linear programming in this paper then turns out to require non-linear programming. Alternatively, the optimization of the lower bounds and the identification of significant upper bound safety margins can be carried out using a representative Gaussian joint distribution, followed by a reliability computation using a general purpose probabilistic analysis program.
- The reliability models considered are formulated in terms of random variables. Generalizations to random process models should be considered.

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